The Edge Spectrum of Maximal k-Colorable Graphs

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Abstract

A graph G on n vertices is *maximal k-colorable* if $\chi(G) = k$ and adding any edge raises the chromatic number. These graphs are exactly the complete k-partite graphs. We study the *edge spectrum* $\Sigma_\chi(n, k)$, the set of edge counts realized by maximal k-colorable graphs on n vertices.

We prove $\mathrm{sat}_\chi(n, k) = (k-1)(2n-k)/2$ and $\mathrm{ab}_\chi(n, k) = \mathrm{ex}(n, K_{k+1})$, and show that $\Sigma_\chi(n, k)$ is a complete integer interval if and only if $n \le k+2$. For $k = 2$ we determine $\Sigma_\chi(n, 2)$ exactly: it has $\lfloor n/2 \rfloor$ elements with gap sizes $n - 2a - 2$. For $k = 3$ we give a parametric description and show that, unlike the bipartite case, distinct partitions can produce the same edge count; the first such collision occurs at $n = 9$. We establish the containment $\Sigma_\chi(n, k) \subseteq \mathrm{ES}(n, K_{k+1})$, which is strict for $n \ge k + 3$, and prove $|\Sigma_\chi(n, k)| \le p_k(n)$. We pose several open problems on the gap structure and collision rates.

Keywords: maximal k-colorable graphs, complete multipartite graphs, edge spectrum, saturation spectrum, Turán graph, extremal graph theory.

1 Introduction

1.1 Definitions

The chromatic number $\chi(G)$ of a graph G is the minimum number of colors in a proper vertex coloring.

Definition 1.1. *A graph G on n vertices is* maximal k-colorable *if $\chi(G) = k$ and $\chi(G+uv) = k+1$ for every non-edge uv.*

Definition 1.2. *For $n \ge k \ge 2$, define the* chromatic saturation number $\mathrm{sat}_\chi(n, k)$*, the* chromatic abundance $\mathrm{ab}_\chi(n, k)$*, the* edge spectrum $\Sigma_\chi(n, k)$*, and the* chromatic saturation width $\Delta_\chi(n, k)$ *by*

$$\begin{aligned}
\mathrm{sat}_\chi(n, k) &= \min\{e(G) : G \text{ is maximal } k\text{-colorable on } n \text{ vertices}\},\\
\mathrm{ab}_\chi(n, k) &= \max\{e(G) : G \text{ is maximal } k\text{-colorable on } n \text{ vertices}\},\\
\Sigma_\chi(n, k) &= \{e(G) : G \text{ is maximal } k\text{-colorable on } n \text{ vertices}\},\\
\Delta_\chi(n, k) &= \mathrm{ab}_\chi(n, k) - \mathrm{sat}_\chi(n, k).
\end{aligned}$$

1.2 Prior work

It is classical that a graph is maximal k-colorable if and only if it is a complete k-partite graph; see Diestel [7]. The saturation number $\mathrm{sat}(n, H)$ of Erdős, Hajnal, and Moon [9] is the minimum number of edges in an H-free graph on n vertices to which no edge can be added without creating H. For $H = K_{k+1}$, they proved $\mathrm{sat}(n, K_{k+1}) = (k - 1)(2n - k)/2$.

Barefoot, Casey, Fisher, Fraughnaugh, and Harary [5] introduced the saturation spectrum $\mathrm{ES}(n, H) = \{e(G) : G \text{ is } H\text{-saturated on } n \text{ vertices}\}$ and determined $\mathrm{ES}(n, K_3)$. Amin, Faudree, and Gould [1] treated $\mathrm{ES}(n, K_4)$; Amin, Faudree, Gould, and Sidorowicz [2] studied $\mathrm{ES}(n, K_p)$ for general p. Faudree, Gould, Jacobson, and Thomas [10] studied gaps in saturation spectra of trees; Balister, Bollobás, Lehel, and Morayne [4] investigated paths and stars. Biró, Horn, and Korándi [6] studied saturation games maintaining k-colorability, observing that graphs saturated with respect to $\chi > k$ are exactly the complete k-partite graphs.

Our $\Sigma_\chi(n, k)$ is the chromatic analogue of $\mathrm{ES}(n, K_{k+1})$: since maximal k-colorable graphs are complete k-partite, it is exactly the set of edge counts of complete k-partite graphs on n vertices.

1.3 Contributions and context

The characterization of maximal k-colorable graphs as complete k-partite graphs is classical, and the extremal edge counts $\mathrm{sat}_\chi(n, k)$ and $\mathrm{ab}_\chi(n, k)$ follow from standard convexity arguments together with Turán's theorem. We include self-contained proofs for completeness. The following results appear to be new:

(i) The interval threshold theorem: $\Sigma_\chi(n, k)$ is a complete integer interval if and only if $n \le k+2$ (Theorem 6.2).

(ii) The first-gap formula: for $n \ge k + 3$, the two smallest spectrum values differ by exactly $n - k - 1$, and no spectrum value lies between them (Lemma 6.1).

(iii) The complete determination of $\Sigma_\chi(n, 2)$ with explicit gap structure (Theorem 6.4).

(iv) The parametric description of $\Sigma_\chi(n, 3)$ and identification of the collision phenomenon (Proposition 7.1 and Theorem 7.3).

(v) The containment $\Sigma_\chi(n, k) \subsetneq \mathrm{ES}(n, K_{k+1})$ for $n \ge k + 3$ (Propositions 9.1–9.2).

1.4 Summary of main results

Extremal parameters (Section 4).

$$\mathrm{sat}_\chi(n, k) = \frac{(k-1)(2n-k)}{2}, \qquad \mathrm{ab}_\chi(n, k) = \mathrm{ex}(n, K_{k+1}) = e(T(n, k)).$$

Width (Section 5). When $k \mid n$: $\Delta_\chi(n, k) = (k-1)(n-k)^2/(2k)$.

Spectrum structure (Section 6).

- $\Sigma_\chi(n, k)$ is gap-free if and only if $n \le k + 2$.
- For $k = 2$: $|\Sigma_\chi(n, 2)| = \lfloor n/2 \rfloor$; consecutive gaps have sizes $n - 2a - 2$.

$k = 3$ **(Section 7).** Distinct partitions of n into 3 parts can yield equal edge counts; the first collision occurs at $n = 9$.

Relation to $\mathrm{ES}(n, K_{k+1})$ **(Section 9).** $\Sigma_\chi(n, k) \subsetneq \mathrm{ES}(n, K_{k+1})$ for $n \ge k + 3$.

2 Notation and Background

All graphs are simple and undirected. For a graph G, write $V(G)$, $E(G)$, $e(G) = |E(G)|$, $\overline{G}$ (complement), and $N(v)$ (open neighborhood). Write K_n, C_n for the complete graph and cycle on n vertices, and $K_{a_1,\ldots,a_k}$ for the complete k-partite graph with part sizes $a_1, \ldots, a_k$. The *Turán graph* $T(n, k)$ is the complete k-partite graph on n vertices with parts as equal as possible.

A *partition of n into k parts* is a k-tuple $(a_1, \dots, a_k)$ of positive integers with $a_1 \le \dots \le a_k$ and $\sum a_i = n$. We write $p_k(n)$ for the number of such partitions.

Theorem 2.1 (Turán [13])**.** *The maximum number of edges in a K_{k+1}-free graph on n vertices is $e(T(n,k))$, achieved uniquely by $T(n,k)$.*

Writing $n = qk + r$ with $0 \le r < k$, the Turán graph has r parts of size $q+1$ and $k-r$ parts of size q, giving

$$e(T(n,k)) = \binom{k}{2} q^2 + r(k-1)q + \binom{r}{2}. \tag{1}$$

Theorem 2.2 (Erdős–Hajnal–Moon [9])**.** *For $n \ge k+1 \ge 3$, $\mathrm{sat}(n, K_{k+1}) = (k-1)n - \binom{k}{2}$, achieved uniquely by $K_{1,\dots,1,n-k+1}$.*

3 The Characterization

Lemma 3.1 (Color-Class Lemma)**.** *Let G be maximal k-colorable and let $uv \notin E(G)$. Then every proper k-coloring of G assigns u and v the same color.*

Proof. If some proper k-coloring c satisfies $c(u) \ne c(v)$, then c remains proper in $G + uv$, so $\chi(G+uv) \le k$, contradicting maximality. □

Theorem 3.2 (Characterization)**.** *A graph G on $n \ge k$ vertices is maximal k-colorable if and only if $G \cong K_{a_1,\dots,a_k}$ for some partition $(a_1, \dots, a_k)$ of n.*

Proof. ($\Leftarrow$) Let $G = K_{a_1,\dots,a_k}$ with parts $V_1, \dots, V_k$. The partition gives a proper k-coloring; since G contains K_k, we have $\chi(G) = k$. For any non-edge uv with $u, v \in V_i$: in any proper coloring of $G + uv$, the vertices u and v receive distinct colors, and each of the $k-1$ remaining parts—being pairwise completely adjacent and adjacent to both u and v—requires a color distinct from the other k already used. Thus $\chi(G+uv) \ge k+1$.

($\Rightarrow$) Fix a proper k-coloring with color classes $S_1, \dots, S_k$. For $u \in S_i$, $v \in S_j$ with $i \ne j$: if $uv \notin E(G)$, then $c(u) \ne c(v)$, contradicting Lemma 3.1. Hence every inter-class pair is an edge, so $G = K_{|S_1|,\dots,|S_k|}$. □

Remark 3.3. *Theorem 3.2 is classical; see Diestel [7, Proposition 5.2.3].*

4 Extremal Parameters

By Theorem 3.2, computing sat_χ and ab_χ reduces to optimizing

$$e(K_{a_1,\dots,a_k}) = \sum_{i<j} a_i a_j = \frac{n^2 - \sum a_i^2}{2} \tag{2}$$

over all partitions of n into k parts.

Lemma 4.1 (Transfer Lemma)**.** *Let $(a_1, \dots, a_k)$ be a composition of n into positive integers with $a_i - a_j \ge 2$ for some i, j. Replacing (a_i, a_j) by $(a_i - 1, a_j + 1)$ strictly decreases $\sum a_\ell^2$; the reverse replacement strictly increases it.*

Proof. $(a_i - 1)^2 + (a_j + 1)^2 - a_i^2 - a_j^2 = -2(a_i - a_j) + 2 \le -2 < 0.$ □

Theorem 4.2 (Saturation Number)**.** *For all* $n \geq k \geq 2$*,*

$$\mathrm{sat}_\chi(n,k) = \frac{(k-1)(2n-k)}{2},$$

achieved uniquely by $K_{1,\dots,1,n-k+1}$*.*

Proof. Minimizing edges is equivalent to maximizing $\sum a_i^2$. By Lemma 4.1, the unique maximizer is $(1,\dots,1,n-k+1)$, with $\sum a_i^2 = (n-k+1)^2 + (k-1)$. Substituting into (2):

$$\mathrm{sat}_\chi(n,k) = \frac{n^2 - (n-k+1)^2 - (k-1)}{2} = \frac{(k-1)(2n-k)}{2}. \qquad \square$$

Corollary 4.3. $\mathrm{sat}_\chi(n,2) = n-1$*;* $\mathrm{sat}_\chi(n,3) = 2n-3$*;* $\mathrm{sat}_\chi(n,n) = \binom{n}{2}$*.*

Theorem 4.4 (Abundance)**.** *For all* $n \geq k \geq 2$*,* $\mathrm{ab}_\chi(n,k) = e(T(n,k)) = \mathrm{ex}(n, K_{k+1})$*, achieved uniquely by* $T(n,k)$*.*

Proof. Maximizing edges is equivalent to minimizing $\sum a_i^2$. By Lemma 4.1, the unique minimizer among compositions of n into k positive parts is the Turán partition. Every complete k-partite graph is K_{k+1}-free, so $\mathrm{ab}_\chi(n,k) \leq \mathrm{ex}(n, K_{k+1})$; equality holds at $T(n,k)$. $\square$

Remark 4.5 (Connection to the Erdős–Hajnal–Moon theorem)**.** *The class of* K_{k+1}*-saturated graphs strictly contains the class of maximal* k*-colorable graphs (for instance,* C_5 *is* K_3*-saturated but not complete bipartite), so* $\mathrm{sat}(n, K_{k+1}) \leq \mathrm{sat}_\chi(n,k)$*. The numerical equality* $\mathrm{sat}(n, K_{k+1}) = \mathrm{sat}_\chi(n,k)$ *holds because* $K_{1,\dots,1,n-k+1}$ *achieves the minimum in both classes.*

5 The Chromatic Saturation Width

Theorem 5.1 (Width)**.**

(a) When $k \mid n$*:* $\Delta_\chi(n,k) = \dfrac{(k-1)(n-k)^2}{2k}$*.*

(b) For fixed k *as* $n \to \infty$*:* $\Delta_\chi(n,k) \sim \dfrac{k-1}{2k}\, n^2$*.*

Proof. For (a), write $n = qk$. Then $\mathrm{ab}_\chi(n,k) = \binom{k}{2} q^2$ and $\mathrm{sat}_\chi(n,k) = (k-1)(2qk-k)/2 = k(k-1)(2q-1)/2$, so

$$\Delta_\chi(n,k) = \frac{k(k-1)}{2}\big(q^2 - (2q-1)\big) = \frac{k(k-1)(q-1)^2}{2} = \frac{(k-1)(n-k)^2}{2k}.$$

Part (b) follows since $\mathrm{sat}_\chi(n,k) = O(kn)$ and $\mathrm{ab}_\chi(n,k) = \frac{k-1}{2k}\, n^2 + O(kn)$. $\square$

Remark 5.2. $\Delta_\chi(n,k) = 0$ *if and only if* $p_k(n) = 1$*, i.e.,* $n \leq k+1$*. (When* $n = k$*, the only partition is* $(1,\dots,1)$*; when* $n = k+1$*, the only partition is* $(1,\dots,1,2)$*.)*

6 The Edge Spectrum

By Theorem 3.2,

$$\Sigma_\chi(n,k) = \left\{ \frac{n^2 - \sum a_i^2}{2} : (a_1,\dots,a_k) \text{ is a partition of } n \text{ into } k \text{ parts} \right\}.$$

Table 1: Extremal parameters for selected values of n and k.

k	n	sat_χ	ab_χ	Δ_χ
2	6	5	9	4
2	10	9	25	16
3	6	9	12	3
3	9	15	27	12
3	12	21	48	27
4	8	18	24	6
4	12	30	54	24

6.1 The interval threshold

Lemma 6.1 (First Gap). *For $n \geq k+1$, the two partitions of n into k parts giving the fewest edges are*

$$C_1 = (1,\ldots,1,n-k+1), \qquad e(C_1) = \mathrm{sat}_\chi(n,k),$$
$$C_2 = (1,\ldots,1,2,n-k), \qquad e(C_2) = \mathrm{sat}_\chi(n,k) + (n-k-1),$$

and no partition achieves an edge count strictly between $e(C_1)$ and $e(C_2)$.

Proof. A direct calculation gives $\sum a_i^2(C_1) - \sum a_i^2(C_2) = 2(n-k-1)$, so $e(C_2)-e(C_1) = n-k-1$. Since C_1 uniquely maximizes $\sum a_i^2$ (Theorem 4.2), and the only way to decrease $\sum a_i^2$ from C_1 while preserving positivity of all parts is to transfer one unit from the part of size $n-k+1$ to a part of size 1 (yielding C_2), every partition other than C_1 and C_2 has $\sum a_i^2 \leq \sum a_i^2(C_2) - 2$, hence edge count $\geq e(C_2)+1$. □

Theorem 6.2 (Interval Threshold). *$\Sigma_\chi(n,k)$ is a complete integer interval if and only if $n \leq k+2$.*

Proof. If $n \geq k+3$, then $e(C_2)-e(C_1) = n-k-1 \geq 2$, and no spectrum value lies between them (Lemma 6.1), so the spectrum has a gap.

For $n = k$, the only partition is $(1,\ldots,1)$; for $n = k+1$, the only partition is $(1,\ldots,1,2)$. In both cases $|\Sigma_\chi| = 1$. For $n = k+2$, there are exactly two partitions: $(1,\ldots,1,3)$ and $(1,\ldots,1,2,2)$, with edge counts differing by 1. □

Example 6.3. $\Sigma_\chi(5,3) = \{7,8\}$ *(interval);* $\Sigma_\chi(6,3) = \{9,11,12\}$ *(gap at 10).*

6.2 Complete spectrum for $k = 2$

Theorem 6.4 (Spectrum for $k=2$). *For all $n \geq 2$:*

(a) $\Sigma_\chi(n,2) = \{a(n-a) : 1 \leq a \leq \lfloor n/2 \rfloor\}$.

(b) All $\lfloor n/2 \rfloor$ values are distinct, and listed in increasing order they are $1(n-1) < 2(n-2) < \cdots < \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil$.

(c) The gap between consecutive values $a(n-a)$ and $(a+1)(n-a-1)$ consists of $n-2a-2$ missing integers.

Proof. Part (a): every partition of n into 2 parts has the form $(a, n-a)$ with $1 \le a \le \lfloor n/2 \rfloor$.

For (b), suppose $a(n-a) = b(n-b)$ with $1 \le a < b \le \lfloor n/2 \rfloor$. Then $(a-b)(n-a-b) = 0$. Since $a \ne b$, we need $a+b=n$, but $a < b \le \lfloor n/2 \rfloor$ forces $a+b<n$, a contradiction. Monotonicity follows from $(a+1)(n-a-1) - a(n-a) = n-2a-1 > 0$.

Part (c): the consecutive difference is $n-2a-1$, leaving $n-2a-2$ integers absent. ☐

Corollary 6.5. *For $k = 2$ and $n \ge 4$, the total number of integers absent from $\Sigma_\chi(n,2)$ in the interval* $[\mathrm{sat}_\chi(n,2),\ \mathrm{ab}_\chi(n,2)]$ *is* $\lfloor n^2/4 \rfloor - \lfloor 3n/2 \rfloor + 2$.

Proof. The interval $[n-1,\ \lfloor n^2/4 \rfloor]$ has $\lfloor n^2/4 \rfloor - n + 2$ integers; subtracting $|\Sigma_\chi(n,2)| = \lfloor n/2 \rfloor$ gives $\lfloor n^2/4 \rfloor - \lfloor 3n/2 \rfloor + 2$. ☐

Example 6.6. $\Sigma_\chi(10,2) = \{9, 16, 21, 24, 25\}$, *with gap sizes* $6, 4, 2, 0$ *and 12 missing integers.*

7 The Edge Spectrum for $k = 3$

A maximal 3-colorable graph on n vertices is $K_{a,b,c}$ with $a+b+c = n$ and $1 \le a \le b \le c$. By (2),

$$e(K_{a,b,c}) = ab + ac + bc = \frac{n^2 - (a^2+b^2+c^2)}{2}.$$

Thus two partitions (a,b,c) and (a',b',c') of n produce the same edge count if and only if $a^2+b^2+c^2 = a'^2+b'^2+c'^2$.

For $k=2$, we proved that distinct partitions always give distinct edge counts (Theorem 6.4(b)). This property fails for $k=3$.

Proposition 7.1 (Collisions for $k = 3$). *Distinct partitions of n into 3 parts can give equal edge counts. The smallest instance is $n = 9$: the partitions $(1,4,4)$ and $(2,2,5)$ both give $e = 24$.*

Proof. $1^2 + 4^2 + 4^2 = 33 = 2^2 + 2^2 + 5^2$. ☐

Remark 7.2. *The collision phenomenon has a clean explanation. For $k = 2$, the constraints $a+b = n$ and $a^2+b^2 = s$ uniquely determine $\{a,b\}$, so the edge count determines the partition. For $k = 3$, the constraints $a+b+c = n$ and $a^2+b^2+c^2 = s$ form an underdetermined system (two equations in three unknowns), so multiple solutions can occur. This is the classical number-theoretic problem of representing an integer as a sum of three positive squares with prescribed total.*

Theorem 7.3 (Spectrum for $k = 3$). *For all $n \ge 3$:*

- *(a)* $\Sigma_\chi(n,3) = \left\{ \frac{n^2-a^2-b^2-(n-a-b)^2}{2} : 1 \le a \le b \le n-a-b \right\}$.
- *(b)* $|\Sigma_\chi(n,3)| \le p_3(n)$, *with equality when the map from partitions to edge counts is injective.*
- *(c) For $n \le 8$ there are no collisions, so $|\Sigma_\chi(n,3)| = p_3(n)$. At $n = 9$ there is exactly one collision, giving $|\Sigma_\chi(9,3)| = 6 < 7 = p_3(9)$.*

Proof. Part (a) restates (2) with $k = 3$. Part (b) is immediate. Part (c) is verified by direct computation; see Table 2. ☐

Remark 7.4. *Computations for $n \le 300$ reveal that collisions for $k = 3$ are not a vanishing phenomenon: the ratio $|\Sigma_\chi(n,3)|/p_3(n)$ does not tend to 1, but instead stabilizes around 0.76 for $n \not\equiv 0 \pmod 3$ and around 0.55 for $n \equiv 0 \pmod 3$. This suggests that edge-count collisions account for a positive fraction of all partitions in the limit, and that the collision rate depends on the residue of n modulo 3. Understanding the precise limiting behavior appears to require tools from analytic number theory.*

Table 2: The spectrum $\Sigma_\chi(n,3)$ for small n.

n	$p_3(n)$	$\lvert\Sigma_\chi\rvert$	coll.	$\Sigma_\chi(n,3)$
3	1	1	0	$\{3\}$
4	1	1	0	$\{5\}$
5	2	2	0	$\{7,8\}$
6	3	3	0	$\{9,11,12\}$
7	4	4	0	$\{11,14,15,16\}$
8	5	5	0	$\{13,17,19,20,21\}$
9	7	6	1	$\{15,20,23,24,26,27\}$
10	8	8	0	$\{17,23,27,28,29,31,32,33\}$

8 Spectrum Size Bounds

Proposition 8.1 (Upper bound)**.** *For all $n \geq k \geq 2$,*

$$|\Sigma_\chi(n,k)| \leq p_k(n).$$

Proof. Each partition yields at most one edge count. □

For $k=2$, the bound is tight: $|\Sigma_\chi(n,2)| = p_2(n) = \lfloor n/2 \rfloor$. For $k \geq 3$, collisions make the bound strict for certain n.

It is classical (see Andrews [3]) that $p_k(n) \sim n^{k-1}/\big(k!\,(k-1)!\big)$ as $n \to \infty$ for fixed k. In particular, $p_2(n) = \lfloor n/2 \rfloor$ and $p_3(n) = \mathrm{round}(n^2/12)$. Thus $|\Sigma_\chi(n,k)| = O(n^{k-1})$ for each fixed k.

Question 8.2. *Is $|\Sigma_\chi(n,k)| = \Theta(n^{k-1})$ for each fixed $k \geq 3$?*

For $k=3$, since the collision ratio $|\Sigma_\chi(n,3)|/p_3(n)$ appears bounded away from zero (Remark 7.4), this is equivalent to asking whether $|\Sigma_\chi(n,3)| = \Theta(n^2)$. The computational evidence strongly supports $|\Sigma_\chi(n,3)| = \Theta(n^2)$, but a proof would require bounding the maximum number of partitions of n into 3 parts sharing a given value of $\sum a_i^2$—a problem in additive combinatorics for which we do not have sharp results.

9 Relation to the Saturation Spectrum

Proposition 9.1 (Containment)**.** *For all $n \geq k \geq 2$, $\Sigma_\chi(n,k) \subseteq \mathrm{ES}(n, K_{k+1})$.*

Proof. Every maximal k-colorable graph G is K_{k+1}-saturated: since $\chi(G) = k$, the graph G is K_{k+1}-free, and for every non-edge uv, $\chi(G+uv) = k+1$ implies $K_{k+1} \subseteq G + uv$. □

Proposition 9.2 (Strict containment)**.** *For $n \geq k+3$ and $k \geq 2$, $\Sigma_\chi(n,k) \subsetneq \mathrm{ES}(n, K_{k+1})$.*

Proof. We exhibit a K_{k+1}-saturated graph on n vertices that is not complete k-partite. For $k=2$: the odd cycle C_5 is K_3-saturated with $e(C_5) = 5$, but $5 \notin \Sigma_\chi(5,2) = \{4,6\}$. For general $k \geq 3$: the join $C_5 \vee K_{k-2}$ on $n = k+3$ vertices is K_{k+1}-free (since C_5 is K_3-free) and K_{k+1}-saturated, but not complete k-partite (since C_5 is not bipartite). For $n > k+3$, replace one vertex of C_5 by an independent set of size $n-k-2$. □

Example 9.3. *For $k=2$, $n=7$: $\Sigma_\chi(7,2) = \{6,10,12\}$. By Barefoot et al. [5], $\mathrm{ES}(7,K_3) = \{6\} \cup [9,12]$. The surplus $\{9,11\}$ is realized by K_3-saturated graphs that are not complete bipartite.*

10 Conjectures and Open Problems

Conjecture 10.1 (Maximum gap)**.** *For each fixed $k \geq 2$ and all $n \geq k+3$, the largest gap in $\Sigma_\chi(n,k)$—that is, the maximum of $s_{i+1} - s_i - 1$ over consecutive spectrum values $s_i < s_{i+1}$—equals $n-k-2$. This maximum is always achieved at the bottom of the spectrum, between $e(C_1) = \mathrm{sat}_\chi(n,k)$ and $e(C_2)$.*

For $k = 2$ this holds: Theorem 6.4(c) shows that gaps decrease monotonically, so the first gap $n-4$ is the largest.

Question 10.2. *What is the limiting value of $|\Sigma_\chi(n,3)|/p_3(n)$ as $n \to \infty$? Computations suggest it depends on $n \bmod 3$: approximately 0.76 for $n \not\equiv 0 \pmod 3$ and approximately 0.55 for $n \equiv 0 \pmod 3$. Can these constants be determined in closed form?*

Question 10.3. *For which n and k do the two largest elements of $\Sigma_\chi(n,k)$ differ by exactly 1?*

11 Spectrum Tables

Table 3: The spectrum $\Sigma_\chi(n,k)$ for small values.

k	n	$\Sigma_\chi(n,k)$
2	4	$\{3, 4\}$
2	5	$\{4, 6\}$
2	6	$\{5, 8, 9\}$
2	7	$\{6, 10, 12\}$
3	5	$\{7, 8\}$
3	6	$\{9, 11, 12\}$
3	7	$\{11, 14, 15, 16\}$
3	8	$\{13, 17, 19, 20, 21\}$
4	6	$\{12, 13\}$
4	7	$\{15, 17, 18\}$
4	8	$\{18, 21, 22, 23, 24\}$

References

[1] K. Amin, J. R. Faudree, and R. J. Gould, The edge spectrum of K_4-saturated graphs, *J. Combin. Math. Combin. Comput.* **81** (2012), 233–242.

[2] K. Amin, J. R. Faudree, R. J. Gould, and E. Sidorowicz, On the non-$(p-1)$-partite K_p-free graphs, *Discuss. Math. Graph Theory* **33** (2013), 9–23.

[3] G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, 1976.

[4] P. Balister, B. Bollobás, J. Lehel, and M. Morayne, On the edge spectrum of saturated graphs for paths and stars, *J. Graph Theory* **89** (2018), 364–380.

[5] C. A. Barefoot, K. F. Casey, D. C. Fisher, K. Fraughnaugh, and F. Harary, Size in maximal triangle-free graphs and minimal graphs of diameter 2, *Discrete Math.* **138** (1995), 93–99.

[6] C. Biró, P. Horn, and D. Korándi, On saturation games, *European J. Combin.* **51** (2016), 315–328.

[7] R. Diestel, *Graph Theory*, 5th ed., Springer-Verlag, Heidelberg, 2017.

[8] D. A. Duffus and D. Hanson, Minimal k-saturated and color critical graphs of prescribed minimum degree, *J. Graph Theory* **10** (1986), 55–67.

[9] P. Erdős, A. Hajnal, and J. W. Moon, A problem in graph theory, *Amer. Math. Monthly* **71** (1964), 1107–1110.

[10] J. Faudree, R. J. Gould, M. S. Jacobson, and B. J. Thomas, Gaps in the saturation spectrum of trees, *Discuss. Math. Graph Theory* **39** (2019), 157–170.

[11] J. Faudree, R. J. Gould, M. S. Jacobson, and B. J. Thomas, Saturation spectrum of paths and stars, *Discuss. Math. Graph Theory* **37** (2017), 811–822.

[12] L. Kászonyi and Zs. Tuza, Saturated graphs with minimal number of edges, *J. Graph Theory* **10** (1986), 203–210.

[13] P. Turán, On an extremal problem in graph theory (Hungarian), *Mat. Fiz. Lapok* **48** (1941), 436–452.